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Analysis of Plane Electrostatic Fields in Doubly Connected Polygonal Domains Using Conformal Mappings

V. V. VECHESLAVOV AND N. I. TOLSTOBROVA

Institute of Nuclear Physics, USSR Academy of Sciences, Novosibirsk 630090

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A complete analysis of an electrostatic field in a doubly connected polygonal domain may be carried out by determining a mapping function which maps an annular ring onto the polygonal domain. This function is well-defined up to a set of unknown parameters, which must be computed. This paper presents a method for computing the true values of these parameters to any desired accuracy, starting from arbitrarily chosen initial values. Computational techniques are discussed and examples are presented which illustrate the applicability of the method.

Let us consider Dirichlet's problem for a doubly connected domain G in the plane of the complex variable $z = x + iy$. The domain may be infinite ($z = \infty \in G$) or finite; it is formed by two polygons with N and $M - N$ vertices, respectively. A concentric annular ring $h \leqslant w \leqslant 1$ in the complex w plane may be single-valuedly and conformally mapped onto this domain. The vertices A_n, $n = 1,..., N$ and A_m, $m = N + 1,..., M$ correspond to the points $a_n = \exp(i\psi_n)$ and $a_m = h \cdot \exp(i\psi_m)$, lying on the boundary circles $|w| = 1$ and $|w| = h$. In the case of the infinite domain, the point $z = \infty$ corresponds to some interior point of the ring $w_\infty = q \cdot \exp(i\varphi)$, $h < q < 1$.

The functions defining this mapping for the infinite and finite domains, respectively, may be written [1]

$$z = C_1 \int \prod_{n=1}^{N} \left[\vartheta_1\left(\frac{1}{2\pi i} \ln \frac{w}{a_n}\right)\right]^{\beta_n} \cdot \prod_{m=N+1}^{M} \left[\vartheta_1\left(\frac{1}{2\pi i} \ln \frac{a_m}{w}\right)\right]^{\beta_m}$$

$$\times \left[\vartheta_1\left(\frac{1}{2\pi i} \ln \frac{w}{w_\infty}\right) \cdot \vartheta_1\left(\frac{\ln(w \cdot w_\infty)}{2\pi i}\right)\right]^{-2} \cdot \frac{dw}{w^2} + C_2, \tag{1}$$

$$z = C_1 \int \prod_{n=1}^{N} \left[\vartheta_1\left(\frac{1}{2\pi i} \ln \frac{w}{a_n}\right)\right]^{\beta_n} \cdot \prod_{m=N+1}^{M} \left[\vartheta_1\left(\frac{1}{2\pi i} \ln \frac{a_m}{w}\right)\right]^{\beta_m} \cdot \frac{dw}{w^2} + C_2. \tag{2}$$

Here $\vartheta_1(v) = 2 \cdot h^{1/4} \cdot \sum_{k=1}^{\infty} (-1)^{k+1} h^{k \cdot (k-1)} \cdot \mathrm{Sin}[(2k-1)\pi v]$ is the ϑ-function of the first type and $\beta_j = \alpha_j - 1$, where α_j are the interior angles, divided by π, of the polygons in the G domain.

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The construction of the function $z(w)$ is equivalent to solving the previously mentioned Dirichlet problem [1]. However, $z(w)$ is known only up to a set of unknown a priori parameters $\psi_1, \ldots, \psi_M$, h, q, φ, which must be computed. We encounter here the doubly connected analog of the well-known Schwartz–Christoffel problem. In [2] a method is described for determining the parameters relating doubly connected domains with symmetrical and similar polygons. Our development of this method allows us to remove certain of the earlier restrictions; in particular, this paper treats doubly connected polygonal domains of a general form.

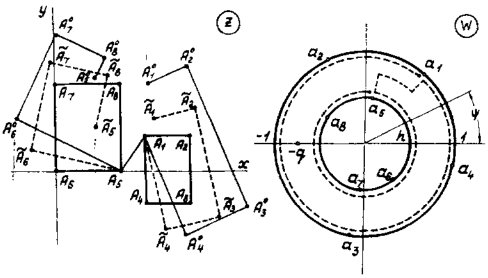

FIGURE 1

First, let us discuss the case of the infinite domain (see Fig. I, where the domain with $N = 4$, $M = 8$ is presented). The interior angles $\pi\alpha_j$ (with respect to the domain G), the lengths of the A_jA_p sides, and the angles $\mu_{j,p}$ of these sides with respect to a real axis are determined by using the given coordinates of the polygonal vertices $A_j = x_j + iy_j$ $(j, p = 1, 2, \ldots, M)$. According to the boundary correspondence principle, the following inequalities are valid:

$$\psi_1 < \psi_2 < \cdots < \psi_N, \qquad \psi_{N+1} > \psi_{N+2} > \cdots > \psi_M. \tag{3}$$

With the help of certain relations for ϑ-functions [1, 3], it is possible to rewrite the basic formula (1) as

$$z = C_1 \int \prod_{n=1}^{N} \left[\vartheta_1\left(\frac{\ln w - i\psi_n}{2\pi i}\right)\right]^{\beta_n} \cdot \prod_{m=N+1}^{M} \left[\vartheta_4\left(\frac{i\psi_m - \ln w}{2\pi i}\right)\right]^{\beta_m}$$
$$\times \left\{\vartheta_1^2\left(\frac{\ln w}{2\pi i}\right) \cdot \vartheta_4^2\left(\frac{\ln w_\infty}{2\pi i}\right) - \vartheta_4^2\left(\frac{\ln w}{2\pi i}\right) \cdot \vartheta_1^2\left(\frac{\ln w_\infty}{2\pi i}\right)\right\}^{-2} \cdot \frac{dw}{iw} + C_2, \tag{4}$$

where $\vartheta_4(v)$ is the ϑ-function of the fourth type.

By integrating around the outer circle $|w| = 1$, $w = \exp(i\psi)$ we obtain, instead of Eq. (4),

$$z = C_1 \int \prod_{n=1}^{N} \left[\vartheta_1\left(\frac{\psi - \psi_n}{2\pi}\right)\right]^{\beta_n} \prod_{m=N+1}^{M} \left[\vartheta_4\left(\frac{\psi_m - \psi}{2\pi}\right)\right]^{\beta_m}$$
$$\times \left\{\vartheta_1^2\left(\frac{\psi}{2\pi}\right) \cdot \vartheta_4^2\left(\frac{\ln w_\infty}{2\pi i}\right) - \vartheta_4^2\left(\frac{\psi}{2\pi}\right) \cdot \vartheta_1^2\left(\frac{\ln w_\infty}{2\pi i}\right)\right\}^{-2} \cdot d\psi + C_2. \tag{5}$$

Similarly, for the inner circle $|w| = h$, $w = h \cdot \exp(i\psi)$,

$$z = C_1 \cdot \exp\left(\frac{i}{2} \sum_{j=1}^{M} \beta_j \cdot \psi_j\right) \cdot \int \prod_{n=1}^{N} \left[\vartheta_4\left(\frac{\psi - \psi_n}{2\pi}\right)\right]^{\beta_n} \cdot \prod_{m=N+1}^{M} \left[\vartheta_1\left(\frac{\psi_m - \psi}{2\pi}\right)\right]^{\beta_m}$$
$$\times \left\{\vartheta_4^2\left(\frac{\psi}{2\pi}\right) \cdot \vartheta_4^2\left(\frac{\ln w_\infty}{2\pi i}\right) - \vartheta_1^2\left(\frac{\psi}{2\pi}\right) \cdot \vartheta_1^2\left(\frac{\ln w_\infty}{2\pi i}\right)\right\}^{-2} \cdot d\psi + C_2 . \tag{6}$$

Under the conformal mapping, the circles $|w| = 1$ and $|w| = h$ correspond to the contours of the polygons in the z plane. Hence, the arguments of z obtained from Eqs. (5) and (6) should undergo jumps by the values $-\pi\beta_j$ at $\psi = \psi_j$ and should not change at the other points. This is possible only with real expressions in the braces over the complete circles $|w| = 1$ and $|w| = h$. We can satisfy this condition by setting

$$\varphi = \pi, \qquad w_\infty = -q, \qquad h < q < 1 \tag{7}$$

and taking into account that $\vartheta_1(v)$, $\vartheta_4(v)$ are real for $v = (\ln q + \pi i)/2\pi i$.

At $\psi_1 < \psi < \psi_2$ and $|w| = 1$ the argument of z obtained from Eq. (5) should be equal to the angle of inclination $\mu_{1,2}$ of the first polygonal side A_1A_2. At $\psi_{N+1} > \psi > \psi_{N+2}$ and $|w| = h$, the argument of z obtained from Eq. (6) should coincide with the angle of inclination $\mu_{N+1,N+2}$ of the second polygonal side $A_{N+1}A_{N+2}$. These requirements lead to the relation

$$\tfrac{1}{2} \cdot \sum_{j=1}^{M} \beta_j \psi_j = \mu_{N+1,N+2} - \mu_{1,2} + \pi \cdot (1 - \beta_1 + \beta_{N+1}). \tag{8}$$

Equation (8) allows one to determine the value of one of the parameters, for instance ψ_1, provided the values $\psi_2, \ldots, \psi_M$ are known. The relations (7), (8) show that in the case of the infinite doubly connected domain the independent mapping parameters which must be determined are the quantities $\psi_2, \ldots, \psi_M$, h, q. It is convenient to treat them as components of some $(M + 1)$-dimensional vector ξ; that is, to set $\xi_1 = \psi_2, \ldots, \xi_{M-1} = \psi_M$, $\xi_M = h$, $\xi_{M+1} = q$.

This paper deals with the procedure for obtaining the true parameter values to any desired accuracy using only a finite number of iterations starting from some initial state ξ^0.

Let us arbitrarily choose initial values $\psi_2^0, \ldots, \psi_M^0$, h^0, q^0 in keeping with Eqs. (3) and (7) and then determine ψ_1^0 from Eq. (8). Put $C_1 = 1$, $C_2 = 0$ and using Eqs. (4), (5), and (6), perform the integration for the contour lying in the w plane as follows (see the dotted line in Fig. I). (1) From the point $w = a_1 = \exp(i\psi_1^0)$ proceed along the complete circle $|w| = 1$. (2) Then proceed along the radius $\psi = \psi_1^0$ to the circle $|w| = (1 + h)/2$, along the segment of this circle to the radius $\psi = \psi_{N+1}^0$, and finally along this radius to the point $w = a_{N+1} = h \cdot \exp(i\psi_{N+1}^0)$. (3) Proceed around the entire circle $|w| = h$. Now we fix the values of the complex scale constants C_1, C_2 in such a way that the starting and the final points of the $z(w)$ curve in the second part of the integration path coincide with the vertices A_1 and A_{N+1}, respectively. Thus, as the result of integration we have a broken line $A_1^0 \cdots A_N^0 A_1 A_{N+1} A_{N+2}^0 \cdots$

$A_M{}^0 A_{N+1}^0$ in the z plane (see the thin line in Fig. I). In the general case, the coordinates $x_j{}^0$, $y_j{}^0$ of the vertex $A_j{}^0$ of this broken line differ from the coordinates x_j, y_j of the corresponding vertex A_j of the true polygon. It is convenient to use a scalar objective function to estimate "the distance" between the initial and the true figures

$$Q(\xi) = (1/M) \cdot \sum_{j=1}^{M} [(x_j - x_j{}^0)^2 + (y_j - y_j{}^0)^2]^{1/2}. \tag{9}$$

If the condition $Q \leqslant Q^* \simeq 0$ is reached, the problem of finding the conformal mapping parameters can be considered solved.

It is necessary to introduce a state which is intermediate with respect to the distance (Eq. (9)) between the initial and the true states [2, 4]. A broken line $\tilde{A}_1 \cdots \tilde{A}_N A_1 A_{N+1} \tilde{A}_{N+2} \cdots \tilde{A}_M \tilde{A}_{N+1}$, where the length $\tilde{A}_j \tilde{A}_p$ of an arbitrary section and the slope $\tilde{\mu}_{j,p}$ of this section are connected with the corresponding lengths $A_j A_p$, $A_j{}^0 A_p{}^0$, and the slopes $\mu_{j,p}$, $\mu_{j,p}^0$ of the true and the initial states are connected by the relations

$$\begin{aligned} \tilde{A}_j \tilde{A}_p &= A_j{}^0 A_p{}^0 + K_\xi \cdot (A_j A_p - A_j{}^0 A_p{}^0), \\ \tilde{\mu}_{j,p} &= \mu_{j,p}^0 + K_\xi \cdot (\mu_{j,p} - \mu_{j,p}^0), \end{aligned} \tag{10}$$

where K_ξ is the intermediate state parameter, satisfies the requirement. Constructed in this way, the broken line for $K_\xi = 1$ coincides with the true figure A and for $K_\xi = 0$ with the initial figure A^0 (the dotted line in Fig. I corresponds to $K_\xi = 0.5$). Let us denote by $\tilde{x}_j$, $\tilde{y}_j$ the coordinates of the vertex $\tilde{A}_j$.

In practice, the computational procedure consists of several stages. In the course of each stage, a transition from the initial state to some intermediate state is made. To determine the correction $\Delta\xi(K_\xi)$ to the value of ξ^0, one solves the system of equations

$$\sum_{j=1}^{M+1} \frac{\partial x_p{}^0}{\partial \xi_j} \cdot \Delta\xi_j = \tilde{x}_p - x_p{}^0, \qquad \sum_{j=1}^{M+1} \frac{\partial y_p{}^0}{\partial \xi_j} \cdot \Delta\xi_j = \tilde{y}_p - y_p{}^0, \tag{11}$$

for $p = 1, 2,..., M$. Since the number of variables here is less than the number of equations, it is natural to obtain the required solution using least squares. Hence, in the linear approach, the optimal corrections are the roots of the normal system

$$\begin{aligned} &\sum_{j=1}^{M+1} \left[\sum_{p=1}^{M} \left(\frac{\partial x_p{}^0}{\partial \xi_j} \cdot \frac{\partial x_p{}^0}{\partial \xi_\nu} + \frac{\partial y_p{}^0}{\partial \xi_j} \cdot \frac{\partial y_p{}^0}{\partial \xi_\nu} \right) \right] \cdot \Delta\xi_j \\ &\quad = \sum_{p=1}^{M} \left[\frac{\partial x_p{}^0}{\partial \xi_\nu} \cdot (\tilde{x}_p - x_p{}^0) + \frac{\partial y_p{}^0}{\partial \xi_\nu} \cdot (\tilde{y}_p - y_p{}^0) \right], \end{aligned} \tag{12}$$

where $\nu = 1, 2,..., M, M + 1$. The calculation is finished when a transition is made to the state $\tilde{A}(\tilde{K}_\xi)$ for which the condition $Q[\xi^1(\tilde{K}_\xi)] < Q(\xi^0)$ is realized. If $Q(\xi^1) > Q^*$, then the state $\tilde{A}$ just determined is taken as the initial state for the next stage, and the process is repeated again. To determine a suitable value of $0 < \tilde{K}_\xi \leqslant 1$, it is convenient to carry out a simple sorting of the K_ξ values which decrease from unity, as is recommended in [2, 4]. These references also show that the system (12) can always be solved and that the reqiured value of $\bar{K}_\xi$ can always be found. The rate of convergence of the computational process depends upon the total number of domain vertices. This dependence needs special study and is not discussed here.

The partial derivatives in Eq. (12) are determined by numerical integration. Therefore, the process of numerical integration for the above-mentioned contour has to be carried out repeatedly during the course of the calculation. The contour integration path passes through the singular points a_j ; in the neighborhood of these points it is impossible to use Eqs. (4), (5), and (6) directly, and transformations of the integrands are required. When integrating around both circles it is necessary only to determine the lengths of the chains of the broken line A^0, because their mutual location is entirely defined by the angles $\pi\alpha_j$ of the true polygons. The computation of these types of integrals may be realized with the help of nonlinear substitutions, and is described in detail in [2].

Under the integral sign along the path between the circles, the starting point $a_1 = \exp(i\psi_1)$ and the final point $a_{N+1} = h \cdot \exp(i\psi_{N+1})$ are singular. Let us choose segments of small length $\Delta \ll 1$, lying on the radii $\psi = \psi_1$, $\psi = \psi_{N+1}$ and adjoining points a_1 and a_{N+1}. Equation (4) can then be used directly everywhere along the path of integration except along these segments. In the neighborhood of the singular point a_1 one can find the value of the integral, using the equation

$$\begin{aligned} I_1(\Delta) = iC_1 \cdot (i\Omega)^{\beta_1} \int_0^\Delta \prod_{n=2}^{N} \left[\vartheta_1\left(\frac{\psi_1 - \psi_n}{2\pi} + \frac{\ln(1-\delta)}{2\pi i}\right)\right]^{\beta_n} \\ \times \prod_{m=N+1}^{M} \left[\vartheta_4\left(\frac{\psi_m - \psi_1}{2\pi} - \frac{\ln(1-\delta)}{2\pi i}\right)\right]^{\beta_m} \\ \cdot \left\{\vartheta_1^2\left(\frac{\psi_1}{2\pi} + \frac{\ln(1-\delta)}{2\pi i}\right)\cdot \vartheta_4^2\left(\frac{\ln q + \pi i}{2\pi i}\right)\right. \\ \left. - \vartheta_4^2\left(\frac{\psi_1}{2\pi} + \frac{\ln(1-\delta)}{2\pi i}\right)\cdot\vartheta_1^2\left(\frac{\ln q + \pi i}{2\pi i}\right)\right\}^{-2} \cdot \frac{\delta^{\beta_1}}{1-\delta}\cdot d\delta. \end{aligned} \tag{13}$$

Similarly, in the neighborhood of the singular point a_{N+1}, we have

$$\begin{aligned} I_2(\Delta) = iC_1 \cdot (h^{-5/4}\cdot\Omega)^{\beta_{N+1}} \int_0^\Delta \prod_{n=1}^{N} \left[\vartheta_1\left(\frac{\psi_{N+1} - \psi_n}{2\pi} + \frac{\ln(h+\delta)}{2\pi i}\right)\right]^{\beta_n} \\ \times \prod_{m=N+2}^{M} \left[\vartheta_4\left(\frac{\psi_m - \psi_{N+1}}{2\pi} - \frac{\ln(h+\delta)}{2\pi i}\right)\right]^{\beta_m} \cdot \left\{\vartheta_1^2\left(\frac{\psi_{N+1}}{2\pi} + \frac{\ln(h+\delta)}{2\pi i}\right)\right. \\ \left. \cdot\vartheta_4^2\left(\frac{\ln q + \pi i}{2\pi i}\right) - \vartheta_4^2\left(\frac{\psi_{N+1}}{2\pi} + \frac{\ln(h+\delta)}{2\pi i}\right)\cdot\vartheta_1^2\left(\frac{\ln q + \pi i}{2\pi i}\right)\right\}^{-2} \\ \cdot \exp\left(-\frac{\delta\cdot\beta_{N+1}}{2h}\right)\cdot\frac{\delta^{\beta_{N+1}}}{h+\delta}\cdot d\delta, \end{aligned} \tag{14}$$

where

$$\Omega = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot (2k - 1) \cdot h^{[4k\cdot(k-1)+1]/4}.$$

To derive the formulas one uses the relation $\vartheta_1(\delta) \simeq 2\pi\Omega \cdot \delta$, which is valid for small, real values of δ. To make the integrands and their first derivatives continuous in Eqs. (13) and (14), it is convenient to use the substitution of variables recommended in [4] of the form

$$t = \delta^{\beta+1}/(\beta + 1) \quad \text{for } \beta < 0, \qquad t = \delta^{\beta}/\beta \quad \text{for } \beta > 0.$$

The sum of the integrals taken along the two circles determines the residue in the point $\omega_\infty = -q$, whose value tends to zero with the imporvement of the values of the mapping parameters.

Let us consider a finite domain, $z = \infty \notin G$. In this case, one polygon is contained entirely inside the other one. After the transformation, the basic formula (2) takes the form

$$z = C_1 \int \prod_{n=1}^{N} \left[\vartheta_1\left(\frac{\ln w - i\psi_n}{2\pi i}\right)\right]^{\beta_n} \cdot \prod_{m=N+1}^{M} \left[\vartheta_4\left(\frac{i\psi_m - \ln w}{2\pi i}\right)\right]^{\beta_m} \cdot \frac{dw}{iw} + C_2 . \qquad (15)$$

It can easily be seen that all the above relations for the infinite domain are valid for the finite domain too, provided the expression in the braces equals unity. The number of independent mapping parameters and the order of the system (12) is equal to the total number of the vertices M. Equation (2) and all its consequences are invariant with respect to substitutions of variables of the form $w = \eta \cdot \exp(i\gamma)$. Hence, the value of parameter ψ_1 can be fixed from the beginning. We may put, for instance, $\psi_1 = 0$. After setting the initial values of the remaining parameters $\xi^0 = (\psi_2{}^0, \ldots, \psi_m{}^0, h^0)$, the procedure is identical to that described above for the infinite domain. In this case the complete integral around both circles is equal to zero, and this fact may be used for the control of computational accuracy.

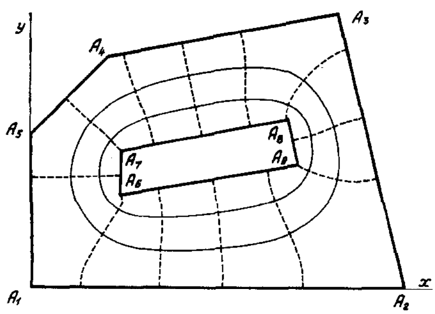

FIGURE 2

Now we consider the finite domain presented in Fig. 2 (the length of the A_1A_2 side is equal to unity) as the first specific example. Domains of this kind appear, for instance, in the design of electrostatic systems for guiding charged particle beams. The initial values of the parameters were arbitrarily taken to be

$$\psi_1{}^0 = 0, \quad \psi_2{}^0 = 1.0, \quad \psi_3{}^0 = 2.0, \quad \psi_4{}^0 = 3.5, \quad \psi_5{}^0 = 5.0,$$

$$\psi_6{}^0 = 0, \quad \psi_7{}^0 = -1.57, \quad \psi_8{}^0 = -3.14, \quad \psi_9{}^0 = -4.71, \quad h^0 = 0.2.$$

We found $Q^0 = 0.2$ using Eq. (9). The final values of these parameters corresponding to $Q < 3 \times 10^{-4}$, $| C_1 | = 0.2664$, and $\arg C_1 = 4.712$ were determined using the procedure discussed earlier. The final values were determined to be

$$\psi_1 = 0, \quad \psi_2 = 1.892, \quad \psi_3 = 3.308, \quad \psi_4 = 4.920, \quad \psi_5 = 5.643,$$

$$\psi_6 = 0.0006, \quad \psi_7 = -0.9206, \quad \psi_8 = -3.122, \quad \psi_9 = -4.100, \quad h = 0.4101.$$

Knowledge of the mapping function defining the conformal mapping carrying the annular ring onto the domain under consideration allows one to perform a detailed analysis for any given distribution of the potential on the domain boundaries [1]. Here we restrict ourselves to the simple case in which the potentials are constant on each polygonal boundary. The calculated mesh of equipotential lines $| w | =$ const and the lines of force $\arg w =$ const are shown in Fig. 2.

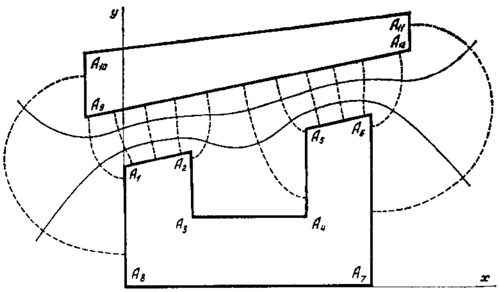

FIGURE 3

The infinite domain employed for the second example is presented in Fig. 3 (the length of the A_7A_8 side is equal to unity). The initial values of the parameters, taking into account Eq. (8), lead to the value $Q^0 = 1.56$ by using the values

$$\psi_1{}^0 = -1.66, \quad \psi_2{}^0 = -1.0, \quad \psi_3{}^0 = 0, \quad \psi_4{}^0 = 1.0, \quad \psi_5{}^0 = 2.0,$$

$$\psi_6{}^0 = 2.50, \quad \psi_7{}^0 = 3.0, \quad \psi_8{}^0 = 3.5, \quad \psi_9{}^0 = -1.0, \quad \psi_{10}^0 = -2.5.$$

$$\psi_{11}^0 = -4.0, \quad \psi_{12}^0 = -6.0, \quad h^0 = 0.2, \quad q^0 = 0.45,$$

The true parameter values correspond to $Q < 10^{-5}$, $|C_1| = 2.572 \times 10^{-3}$, and $\arg C_1 = -4.712$. These values are

$$\psi_1 = -2.021, \quad \psi_2 = -0.5966, \quad \psi_3 = -0.2109, \quad \psi_4 = -0.0084, \quad \psi_5 = 0.5190,$$
$$\psi_6 = 1.998, \quad \psi_7 = 2.939, \quad \psi_8 = 3.431, \quad \psi_9 = -2.355, \quad \psi_{10} = -2.838,$$
$$\psi_{11} = -3.482, \quad \psi_{12} = -3.832, \quad h = 0.3933, \quad q = 0.6285,$$

Domains similar to the one presented in Fig. 3 appear, for instance, in magnetic field calculations in electrodynamic experiments. If the scalar magnetic potentials are constant on the domain boundaries, then the equipotential lines and the lines of flux are the images of the curves $|w| = \text{const}$ and $\arg w = \text{const}$, respectively. The calculated mesh of these lines is presented in Fig. 3.

It is noteworthy that the method discussed here can be advantageous also in those cases when the doubly connected domains are confined by superconductors or by perfect ferromagnetics and when the domains contain free electrostatic charges and currents.

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